Sinusoids and Phasors

Recall that, for capacitors and inductors, the branch variables (current values and voltage values) are related by differential equations. Normally, to analyze a circuit containing capacitor and/or inductor, we need to solve some differential equations. The analysis can be greatly simplifies when the circuit is driven (or excited) by a source (or sources) that is sinusoidal. Such assumption will be the main focus of this chapter.

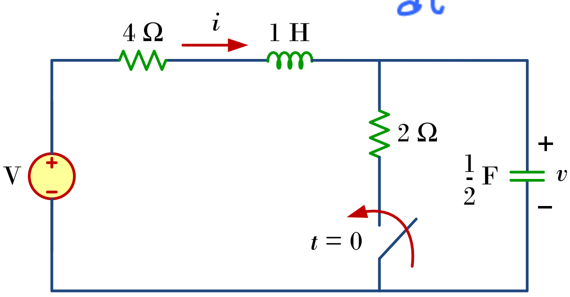
7.1. Prelude to Second-Order Circuits

The next example demonstrates the complication normally involved when analyzing a circuit containing capacitor and inductor. This example and the analysis presented is not the main focus of this chapter.

EXAMPLE 7.1.1. The switch in the figure below has been open for a long time. It is closed at $t = 0$.

$v = L\frac{di}{dt}$

$v_s(t) =$



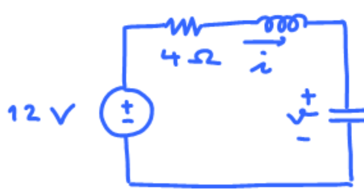
$i = C\frac{dv}{dt}$

(a) Find $v(0)$ and $\frac{dv}{dt}(0)$.
(b) Find $v(t)$ for $t > 0$.
(c) Find $v(\infty)$ and $\frac{dv}{dt}(\infty)$.

The source is $v_s(t) \equiv 12$ V

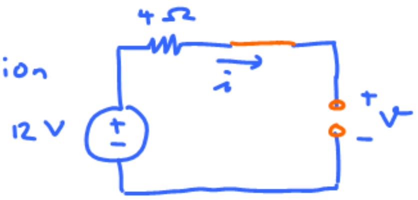
(d) Find $v(t)$ for $t > 0$ when the source is $v_s(t) = \begin{cases} 12, & t < 0, \\ 12\cos(t), & t \geq 0. \end{cases}$

(a) For $t < 0$,



The circuit is left in this configuration for a long time and the driving element is constant (12 V).

dc condition $\Longrightarrow$

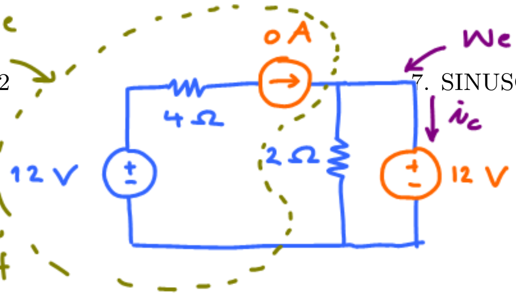


$i(t) = 0$ A

$v(t) = 12$ V

At $t=0$, $i(0) = 0$ A (no jump in i through L)
$v(0) = 12$ V (no jump in v across C)

Due to the 0 A source which is equivalent to an open circuit, this part of the circuit has no current and hence it is just a hanging branch.



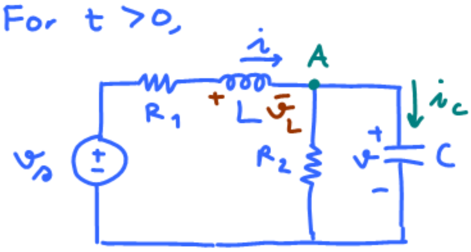
We only have to focus on the loop on the RHS.
Note that $i_C = \frac{-12}{2} = -6$ A.

Also, $i_C = C\frac{dv}{dt}$.

Therefore, $\frac{dv}{dt} = \frac{1}{C} i_C = \frac{-6}{1/2} = -12$ V/s.

$\frac{dv}{dt}(0) = -12$ V/s.

(b) For $t > 0$,



KCL @ A: $-i + \frac{v}{R_2} + i_C = 0$ $\qquad i_C = C\frac{dv}{dt}$

$\Rightarrow i = \frac{v}{R_2} + i_C = \frac{v}{R_2} + Cv'$

KVL around the outer-loop:

$+v_s - iR_1 - v_L - v = 0$ $\qquad v_L = L\frac{di}{dt}$

$\Rightarrow +v_s - iR_1 - Li' - v = 0$

$v_s - \left(\frac{v}{R_2} + Cv'\right)R_1 - L\left(\frac{v}{R_2} + Cv'\right)' - v = 0$

From MATLAB,

$v'' + \left(\frac{1}{CR_2} + \frac{R_1}{L}\right)v' + \frac{1}{LC}\left(1 + \frac{R_1}{R_2}\right)v = \frac{v_s}{LC}$

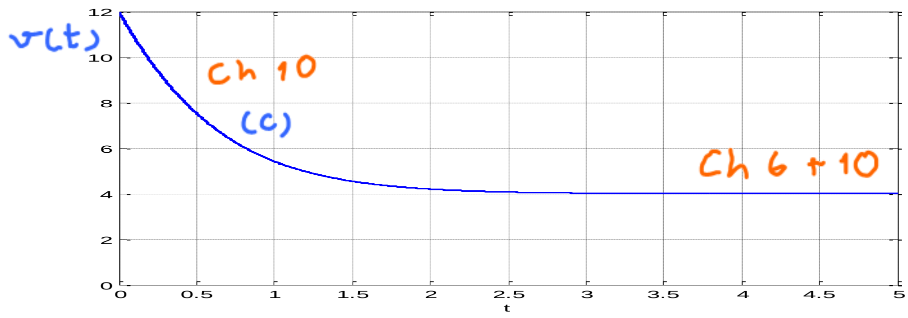
```
v = dsolve('D2v + 5*Dv + 6*v = 24','v(0) = 12','Dv(0) = -12')
```

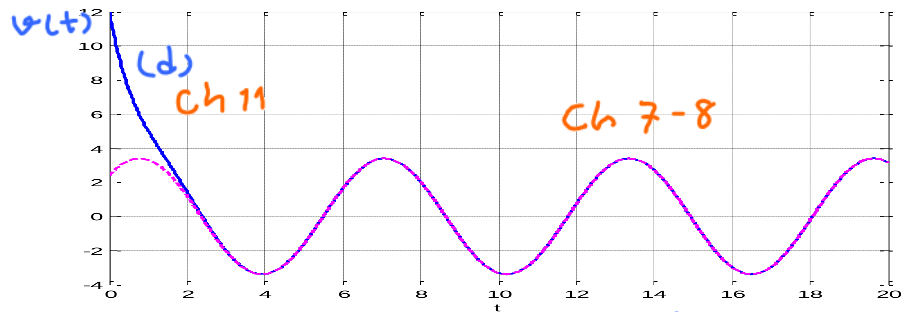
gives $v(t) = 4 + 12e^{-2t} - 4e^{-3t}$. Similarly,

(c) $v(\infty) = 4$, $\frac{dv}{dt}(\infty) = 0$

(d)
```
v = dsolve('D2v + 5*Dv + 6*v = 2*12*cos(t)','v(0) = 12','Dv(0) = -12','t')
```

gives $v(t) = \frac{72}{5}e^{-2t} - \frac{24}{5}e^{-3t} + \frac{12}{5}\cos(t) + \frac{12}{5}\sin(t)$, $t > 0$





$\frac{d^2}{dt^2}v + 5\frac{dv}{dt} + 6v = 24\cos t$

$\Leftrightarrow (j\omega)^2\vec{V} + 5j\omega\vec{V} + 6\vec{V} = 24\angle 0°$ $\qquad \omega = 1$

$-\vec{V} + 5j\vec{V} + 6\vec{V} = 24$

$\vec{V} = \frac{24}{5+5j} = \frac{12\sqrt{2}}{5}\angle -45°$

$\Leftrightarrow v(t) = \frac{12\sqrt{2}}{5}\cos(t - 45°)$

7.2. Sinusoids

DEFINITION 7.2.1. Some terminology:

(a) A **sinusoid** is a signal (, e.g. voltage or current) that has the form of the sine or cosine function. (with 0 average)
  - Turn out that you can express them all under the same notation using only cosine (or only sine) function.
    - We will use cosine.

(b) A sinusoidal current is referred to as **alternating current (AC)**.

(c) We use the term **AC source** for any device that supplies a sinusoidally varying voltage (potential difference) or current.

(d) Circuits driven by sinusoidal current or voltage sources are called **AC circuits**.

angular freq. freq.

7.2.2. Consider the sinusoidal signal (in cosine form)

current $i(t)$, voltage $v(t)$

$$x(t) = X_m \cos(\omega t + \phi) = X_m \cos(2\pi f t + \phi),$$

amplitude phase (angle)

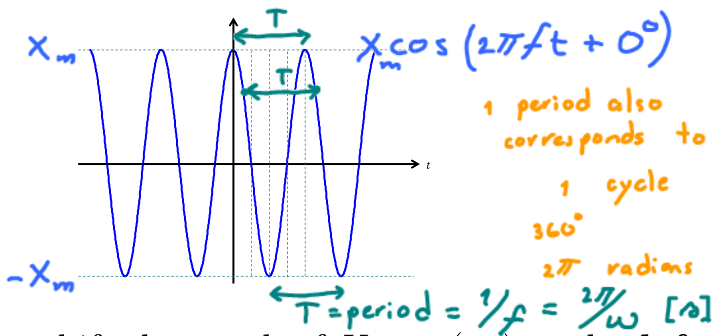
where

X_m: the amplitude of the sinusoid,

$2\pi f =$ ω: the angular frequency in radians/s (or rad/s),

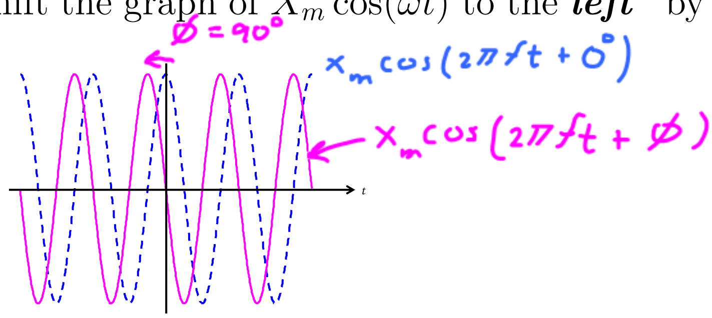
ϕ: the phase.

- First, we consider the case when $\phi = 0$:



$\omega = 2\pi f$
$f = \frac{\omega}{2\pi}$

- When $\phi \neq 0$, we shift the graph of $X_m \cos(\omega t)$ to the ***left*** "by ϕ".



7.2.3. The **period** (the time of one complete cycle) of the sinusoid is

$$T = \frac{2\pi}{\omega}.$$

The unit of the period is in second if the angular frequency unit is in radian per second.

The **frequency** f (the number of cycles per second or hertz (Hz)) is the reciprocal of this quantity, i.e.,

$$f = \frac{1}{T}.$$

7.2.4. ***Standard form*** for sinusoid: In this class, when you are asked to find the sinusoid representation of a signal, make sure that your answer is in the form

$$x(t) = X_m \cos(\omega t + \phi) = X_m \cos(2\pi f t + \phi),$$

where X_m is nonnegative and ϕ is between $-180°$ and $+180°$.

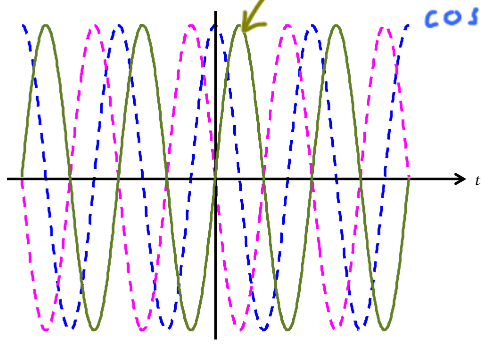
7.2.5. Conversions to standard form

- When the signal is given in the sine form, it can be converted into its cosine form via the identity $= \cos(x + (-90°))$ ≡ cos "shifted to the left by $-90°$" ≡ cos "shifted to the right by $90°$"

$$\sin(x) = \cos(x - 90°).$$

sin(t)

cos(t)



In particular,

$$X_m \sin(\omega t + \phi) = X_m \cos(\omega t + \phi - 90°).$$

- X_m is always non-negative. We can avoid having the negative sign by the following conversion:

$$-\cos(x) = \cos(x \pm 180°).$$

Ex. $-\cos(x - 70°) = \cos(x - 70° \pm 180°)$ → $\cos(x + 110°)$

→ $\cancel{\cos(x - 250°)}$ not in standard form

In particular,

$$-A\cos(\omega t+\phi) = A\cos(2\pi f t+\phi \pm 180°).$$

Note that usually you do not have the choice between $+180°$ or $-180°$. The one that you need to use is the one that makes $\phi \pm 180°$ falls somewhere between $-180°$ and $+180°$.

(eventually)

7.2.6. For any[1] linear AC circuit, the "steady-state" voltage and current are sinusoidal with the same frequency as the driving source(s).

- Although all the voltage and current are sinusoidal, their amplitudes and phases can be different.
  - These can be found by the technique discussed in this chapter.

7.3. Phasors

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions. The tradeoff is that phasors are complex-valued.

$j = \sqrt{-1}$

7.3.1. The idea of phasor representation is based on Euler's identity:

$$e^{j\phi} = \cos\phi + j\sin\phi,$$

ϕ is any real number

From the identity, we may regard $\cos\phi$ and $\sin\phi$ as the real and imaginary parts of $e^{j\phi}$:

$$\cos\phi = \mathrm{Re}\left\{e^{j\phi}\right\}, \quad \sin\phi = \mathrm{Im}\left\{e^{j\phi}\right\},$$

where Re and Im stand for "the real part of" and "the imaginary part of" $e^{j\phi}$.

Definition 7.3.2. A ***phasor*** is a complex number that represents the amplitude and phase of a sinusoid. Given a sinusoid $x(t) = X_m\cos(\omega t+\phi)$, then

$$x(t) = X_m\cos(\omega t+\phi) = \mathrm{Re}\left\{X_m e^{j(\omega t+\phi)}\right\} = \mathrm{Re}\left\{X_m e^{j\phi}\cdot e^{j\omega t}\right\} = \mathrm{Re}\left\{\mathbf{X} e^{j\omega t}\right\},$$

$\vec{X}$ $\underset{\sim}{X}$

where

$$\mathbf{X} = X_m e^{j\phi} = X_m\angle\phi.$$

The complex number $\mathbf{X}$ is called the ***phasor representation*** of the sinusoid $v(t)$. Notice that a phasor captures information about amplitude and phase of the corresponding sinusoid.

Ex. $x(t) = \sqrt{2}\cos(10\pi t + 45°) \Leftrightarrow \vec{X} = \sqrt{2}\angle 45°$

[1]When there are multiple sources, we assume that all sources are at the same frequency.

7.3.3. Whenever a sinusoid is expressed as a phasor, the term $e^{j\omega t}$ is **implicit**. It is therefore important, when dealing with phasors, to keep in mind the frequency f (or the angular frequency ω) of the phasor.

7.3.4. Given a phasor $\mathbf{X}$, to obtain the time-domain sinusoid corresponding to a given phasor, there are two important routes.

(a) Simply write down the cosine function with the same magnitude as the phasor and the argument as ωt plus the phase of the phasor.
(b) Multiply the phasor by the time factor $e^{j\omega t}$ and take the real part.

7.3.5. Any complex number z (including any phasor) can be equivalently represented in three forms.

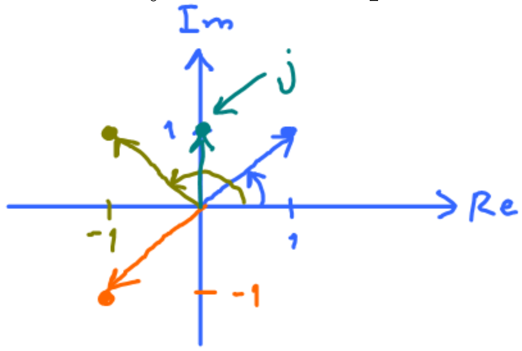
(a) Rectangular form: $z = x + jy$.
(b) Polar form: $z = r\angle\phi$.
(c) Exponential form: $z = re^{j\phi}$

where the relations between them are

$$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1}\frac{y}{x} \pm 180^\circ.$$

$$x = r\cos\phi, \quad y = r\sin\phi.$$

Note that for ϕ, the choice of using $+180^\circ$ or -180° in the formula is determined by the actual quadrant in which the complex number lies.



$\vec{z}$

rectangular form	polar form
$1+j$	$\sqrt{2}\angle 45^\circ$
$-1+j$	$\sqrt{2}\angle 135^\circ$
$-1-j$	$\sqrt{2}\angle -135^\circ$
j	$1\angle 90^\circ$

As a complex quantity, a phasor may be expressed in rectangular form, polar form, or exponential form. In this class, we focus on polar form.

7.3.6. ***Summary***: By suppressing the time factor, we transform the sinusoid from the time domain to the phasor domain. This transformation is summarized as follows:

Same convention
$X_m \geq 0$
$-180^\circ < \phi \leq 180^\circ$

$$x(t) = X_m\cos(\omega t + \phi) \Leftrightarrow \mathbf{X} = X_m\angle\phi.$$

Time domain representation	$\Leftrightarrow$	Phasor domain representation
Real-valued Function of time (time-dependent)		A complex number
Give instantaneous value of the signal at time t.		Frequency domain

Definition 7.3.7. ***Standard form*** for phasor: In this class, when you are asked to find the phasor representation of a signal, make sure that your answer is a complex number in polar form, i.e. $r\angle\phi$ where r is nonnegative and ϕ is between $-180°$ and $+180°$.

Example 7.3.8. Transform these sinusoids to phasors:

(a) $i = 6\cos(50t - 40°)$ A $\Leftrightarrow$ $\vec{I} = 6\angle -40°$

(b) $v = -4\sin(30t + 50°)$ V $= -4\cos(30t + 50° - 90°)$

$= -4\cos(30t - 40°) = 4\cos(30t - 40° \overset{?}{\pm} 180°)$

$= 4\cos(30t + 140°)$ $\Leftrightarrow$ $\vec{V} = 4\angle 140°$

Example 7.3.9. Find the sinusoids represented by these phasors:

(a) $\mathbf{I} = -3 + j4$ A $\underset{\uparrow \text{calc}}{=}$ $5\angle 127°$ $\Leftrightarrow$ $i(t) = 5\cos(\omega t + 127°)$

(b) $\mathbf{V} = j8e^{-j20°}$ V $= (1\angle 90°)(8\angle -20°) = 8\angle 70°$ $\Leftrightarrow$ $v(t) = 8\cos(\omega t + 70°)$

$e^{j90°} \times 8 \times e^{-j20°}$

$(r_1\angle\theta_1)(r_2\angle\theta_2) = r_1 r_2\angle(\theta_1 + \theta_2)$

7.3.10. The differences between $x(t)$ and $\mathbf{X}$ should be emphasized:

(a) $x(t)$ is the instantaneous or time-domain representation, while $\mathbf{X}$ is the frequency or phasor-domain representation.

(b) $x(t)$ is time dependent, while $\mathbf{X}$ is not.

(c) $x(t)$ is always real with no complex term, while $\mathbf{X}$ is generally complex.

7.3.11. Adding sinusoids of the *same frequency* is equivalent to adding their corresponding phasors. To see this,

$$
\begin{aligned}
A_1\cos(\omega t + \phi_1) + A_2\cos(\omega t + \phi_2) &= \operatorname{Re}\left\{\mathbf{A}_1 e^{j\omega t}\right\} + \operatorname{Re}\left\{\mathbf{A}_2 e^{j\omega t}\right\} \\
&= \operatorname{Re}\left\{(\mathbf{A}_1 + \mathbf{A}_2)\, e^{j\omega t}\right\}.
\end{aligned}
$$

Because $\mathbf{A}_1 + \mathbf{A}_2$ is just another complex number, we can conclude a (surprising) fact: adding two sinusoids of the same frequency gives another sinusoids.

Ex. $\cos(t) + \overbrace{\sin(t)}^{\cos(t-90°)}$

$\Leftrightarrow$ $1\angle 0° + 1\angle -90° = \sqrt{2}\angle -45°$

$\Leftrightarrow$ $\sqrt{2}\cos(1t - 45°)$

$\omega = 2$

$= 4\cos(2t + 0^\circ) + 3\cos(2t - 90^\circ)$

EXAMPLE 7.3.12. $x(t) = 4\cos(2t) + 3\sin(2t)$

Direct calculation:

$4\cos(2t + 0^\circ) + 3\cos(2t - 90^\circ)$

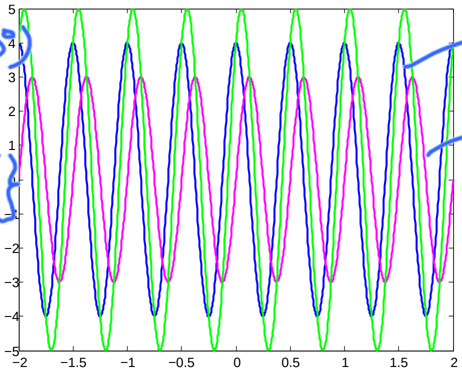
$= \mathrm{Re}\{(4\angle 0^\circ) e^{j2t}\}$

$+ \mathrm{Re}\{(3\angle -90^\circ) e^{j2t}\}$

$= \mathrm{Re}\{((4\angle 0^\circ) + (3\angle -90^\circ)) e^{j2t}\}$

$= \mathrm{Re}\{(5\angle -36.87^\circ) e^{j2t}\}$

$= 5\cos(2t - 36.87^\circ)$



$4\cos(2t)$

$3\cos(2t - 90^\circ)$

$\Leftrightarrow (4\angle 0^\circ) + (3\angle -90^\circ)$
$= 5\angle -36.87^\circ$

$\Leftrightarrow 5\cos(2t - 36.87^\circ)$

7.3.13. Properties involving differentiation and integration:

(a) **Differentiating** a sinusoid is equivalent to multiplying its corresponding phasor by $j\omega$. In other words,

$$\frac{dx(t)}{dt} \Leftrightarrow j\omega\mathbf{X}.$$

To see this, suppose $x(t) = X_m \cos(\omega t + \phi)$. Then,

$$\begin{aligned}\frac{dx}{dt}(t) &= -\omega X_m \sin(\omega t + \phi) = \omega X_m \cos(\omega t + \phi - 90^\circ + 180^\circ) \\ &= \mathrm{Re}\left\{\omega X_m e^{j\phi} e^{j90^\circ} \cdot e^{j\omega t}\right\} = \mathrm{Re}\left\{j\omega \mathbf{X} e^{j\omega t}\right\}\end{aligned}$$

Alternatively, express $v(t)$ as

$$x(t) = \mathrm{Re}\left\{X_m e^{j(\omega t + \phi)}\right\}.$$

Then,

$$\frac{d}{dt}x(t) = \mathrm{Re}\left\{X_m j\omega e^{j(\omega t + \phi)}\right\}.$$

(b) **Integrating** a sinusoid is equivalent to dividing its corresponding phasor by $j\omega$. In other words,

$$\int x(t)dt \Leftrightarrow \frac{\mathbf{X}}{j\omega}.$$

EXAMPLE 7.3.14. Find the voltage $v(t)$ in a circuit described by the intergrodifferential equation

$$2\frac{dv}{dt} + 5v + 10\int v dt = 50\cos(5t - 30^\circ)$$

$\omega = 5$

$\frac{2}{j} \times \frac{j}{j} = \frac{2j}{-1} = -2j$

using the phasor approach.

$\Leftrightarrow \quad 2j\omega\vec{V} + 5\vec{V} + 10\frac{\vec{V}}{j\omega} = 50\angle -30^\circ$

(phasor domain)

$\vec{V}(10j + 5 + \frac{2}{j}) = 50\angle -30^\circ$

$\vec{V} = \frac{50\angle -30^\circ}{10j + 5 - 2j}$

$= 5.3\angle -88^\circ$

$\Leftrightarrow v(t) = 5.3\cos(5t - 88^\circ)$

"steady-state"

7.4. Phasor relationships for circuit elements

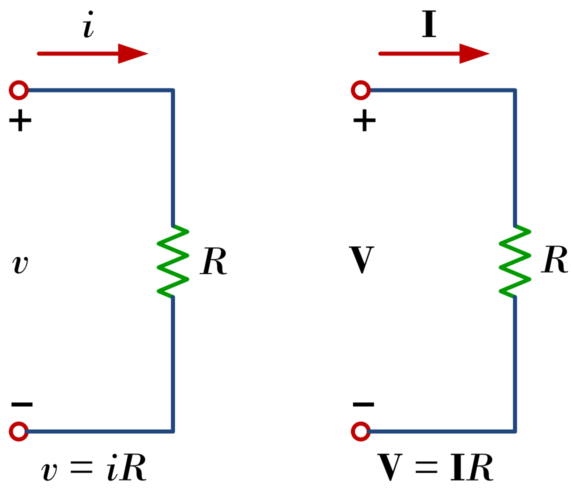
R
C
L

7.4.1. Resistor R: If the current through a resistor R is

$$i(t) = I_m \cos(\omega t + \phi) \Leftrightarrow \mathbf{I} = I_m \angle \phi,$$

the voltage across it is given by Ohm's law

$$v(t) = i(t)R = RI_m \cos(\omega t + \phi).$$



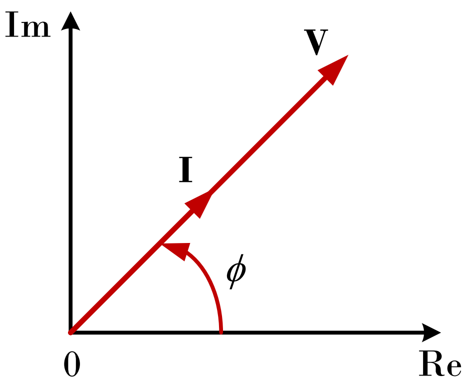
The phasor of the voltage is

$$\mathbf{V} = RI_m \angle \phi.$$

Hence,

$$\mathbf{V} = \mathbf{I}R.$$

We note that voltage and current are **in phase** and that the voltage-current relation for the resistor in the phasor domain continues to be Ohms law, as in the time domain.

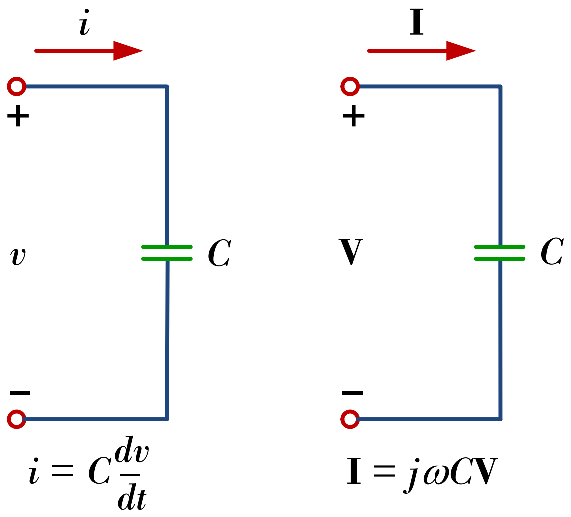


7.4.2. Capacitor C: If the voltage across a capacitor C is

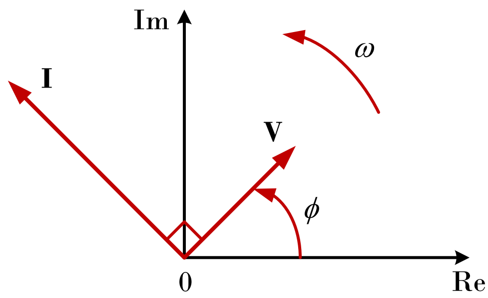
$$v(t) = V_m \cos(\omega t + \phi) \Leftrightarrow \mathbf{V} = V_m \angle \phi,$$

the current through it is given by

$$i(t) = C\frac{dv(t)}{dt} \Leftrightarrow \mathbf{I} = j\omega C\mathbf{V} = \omega C V_m \angle(\phi + 90°).$$



The voltage and current are 90° out of phase. Specifically, the current leads the voltage by 90°.



- Mnemonic: CIVIL

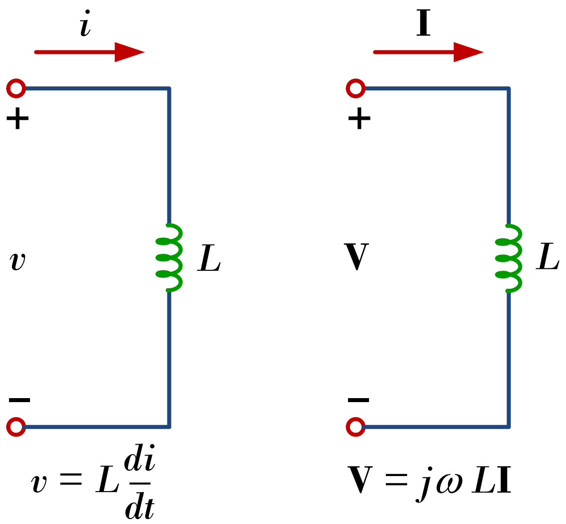
  In a Capacitive (C) circuit, I leads V. In an inductive (L) circuit, V leads I.

7.4.3. Inductor L: If the current through an inductor L is

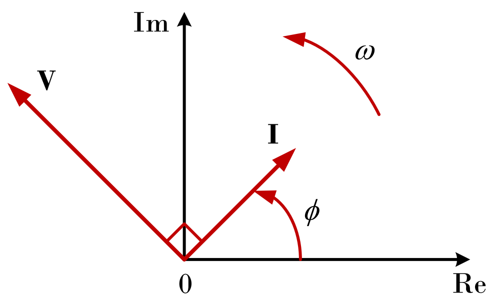
$$i(t) = I_m \cos(\omega t + \phi) \Leftrightarrow \mathbf{I} = I_m \angle \phi,$$

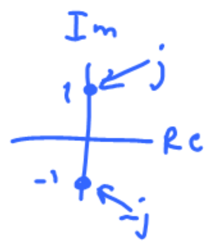
the voltage across it is given by

$$v(t) = L\frac{di(t)}{dt} \Leftrightarrow \mathbf{V} = j\omega L\mathbf{I} = \omega L I_m \angle(\phi + 90^\circ).$$



The voltage and current are 90° out of phase. Specifically, the current lags the voltage by 90°.





Element	Time domain	Frequency domain	Z
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$	R
L	$v = L\frac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$ $\quad 1\angle 90^\circ$	$j\omega L$
C	$i = C\frac{dv}{dt}$	$\mathbf{V} = \frac{\mathbf{I}}{j\omega C}$	$\frac{1}{j\omega C} = -\frac{1}{\omega C}j$

$(-j) = 1$

7.5. Impedance and Admittance

In the previous part, we obtained the voltage current relations for the three passive elements as

$$\mathbf{V} = \mathbf{I}R, \quad \mathbf{V} = j\omega L\mathbf{I}, \quad \mathbf{I} = j\omega C\mathbf{V}.$$

These equations may be written in terms of the ratio of the phasor voltage to the phasor of current as

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \quad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \quad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}. \times \frac{j}{j} = j\left(-\frac{1}{\omega C}\right)$$

From these equations, we obtain Ohm's law in phasor form for any type of element as

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \quad \text{or} \quad \mathbf{V} = \mathbf{I}\mathbf{Z}.$$

DEFINITION 7.5.1. The impedance $\mathbf{Z}$ of a circuit is the ratio of the phasor voltage $\mathbf{V}$ to the phasor current $\mathbf{I}$, measured in ohms (Ω).

As a complex quantity, the impedance may be expressed in rectangular form as

$$\mathbf{Z} = R + jX = \cancel{|\mathbf{Z}|\angle\theta}, \quad = \cancel{|\vec{Z}|e^{j\theta}}$$

with

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \quad \theta = \tan^{-1}\frac{X}{R}, \quad R = |\mathbf{Z}|\cos\theta, \quad X = |\mathbf{Z}|\sin\theta.$$

$R = \operatorname{Re}\{\mathbf{Z}\}$ is called the **resistance** and $X = \operatorname{Im}\{\mathbf{Z}\}$ is called the **reactance**.

The reactance X may be positive or negative. We say that the impedance is **inductive** when X is positive or **capacitive** when X is negative.

DEFINITION 7.5.2. The **admittance** ($\mathbf{Y}$) is the reciprocal of impedance, measured in Siemens (S). The admittance of an element(or a circuit) is the ratio of the phasor current through it to phasor voltage across it, or

$$\mathbf{Y} = \frac{\mathbf{1}}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}}.$$

$\vec{V} = \vec{I}\vec{Z}$

① "Ohm's law":

7.5.3. **Kirchhoff's laws (KCL and KVL) hold in the phasor form.**

To see this, suppose $v_1, v_2, \dots, v_n$ are the voltages around a closed loop, then

$$v_1 + v_2 + \cdots + v_n = 0.$$

If each voltage v_i is a sinusoid, i.e.

$$v_i = V_{mi}\cos(\omega t + \phi_i) = \text{Re}\left\{\mathbf{V}_i e^{j\omega t}\right\}$$

with phasor $\mathbf{V}_i = V_{mi}\angle\phi_i = V_{mi}e^{j\phi_i}$, then

$$\text{Re}\left\{\left(\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n\right)e^{j\omega t}\right\} = 0,$$

which must be true for all time t. To satisfy this, we need

② KVL

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0.$$

Hence, KVL holds for phasors.

Similarly, we can show that KCL holds in the frequency domain, i.e., if the currents $i_1, i_2, \dots, i_n$ are the currents entering or leaving a closed surface at time t, then

$$i_1 + i_2 + \cdots + i_n = 0.$$

If the currents are sinusoids and $\mathbf{I}_1, \mathbf{I}_2, \dots, \mathbf{I}_n$ are their phasor forms, then

③ KCL

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0.$$

7.5.4. **Major Implication:** Since Ohm's Law and Kirchoff's Laws hold in phasor domain, **all resistance combination formulas, volatge and current divider formulas, analysis methods** (nodal and mesh analysis) **and circuit theorems** (linearity, superposition, source transformation, and Thevenin's and Norton's equivalent circuits) that we have previously studied for dc circuits **apply to ac circuits** !!!

Just think of impedance as a complex-valued resistance!!

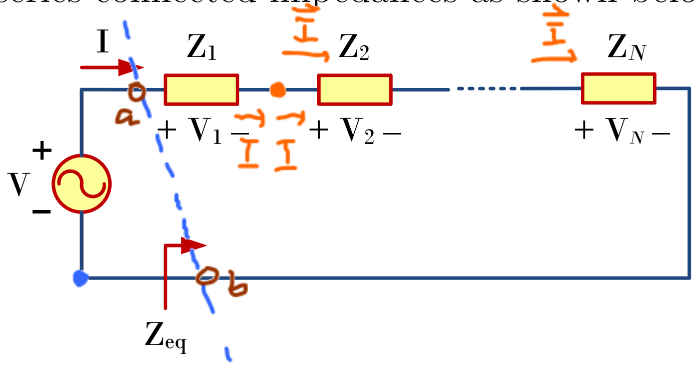
The three-step analysis in the next chapter is based on this insight.

In addition, our ac circuits can now effortlessly include capacitors and inductors which can be considered as impedances whose values depend on the frequency ω of the ac sources!!

$\vec{V} - \vec{V}_1 - \vec{V}_2 - \cdots - \vec{V}_N = 0$

7.6. Impedance Combinations

Consider N series-connected impedances as shown below.



The same current $\mathbf{I}$ flows through the impedances. Applying KVL around the loop gives

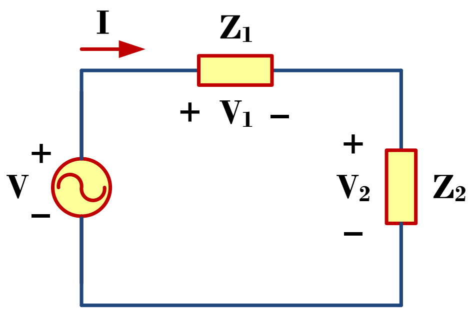
$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_N = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N)$$

The equivalent impedance at the input terminals is

at terminals a-b

$$\mathbf{Z}_{eq} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N.$$

In particular, if $N = 2$, the current through the impedance is



$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2}.$$

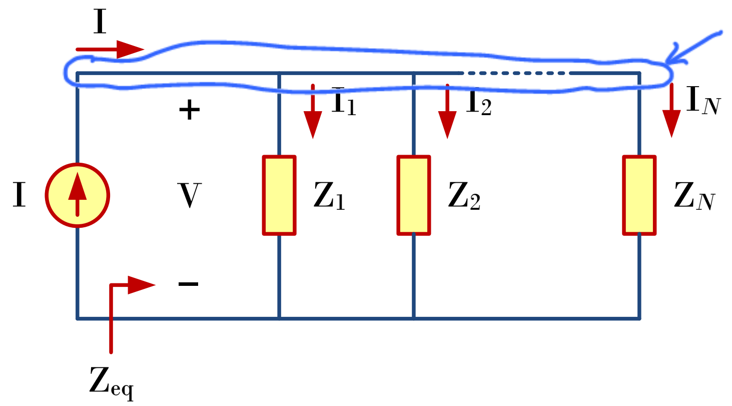
Because $\mathbf{V}_1 = \mathbf{Z}_1\mathbf{I}$ and $\mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}$,

$$\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}, \quad \mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}$$

which is the **voltage-division** relationship.

In general, $\vec{V}_k = \dfrac{\vec{Z}_k}{\vec{Z}_1 + \vec{Z}_2 + \cdots + \vec{Z}_N}\vec{V}$

Now, consider N parallel-connected impedances as shown below.



Apply KCL here

The voltage across each impedance is the same. Applying KCL at the top node gives

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}\right).$$

The equivalent impedance $\mathbf{Z}_{eq}$ can be found from

$$\frac{1}{\mathbf{Z}_{eq}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}.$$

$$\vec{Z}_{eq} = \frac{1}{\frac{1}{\vec{Z}_1} + \frac{1}{\vec{Z}_2} + \cdots + \frac{1}{\vec{Z}_N}}$$

When $N = 2$,

$$\mathbf{Z}_{eq} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}.$$

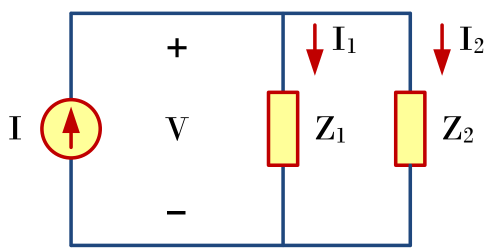
Because

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{eq} = \mathbf{I}_1\mathbf{Z}_1 = \mathbf{I}_2\mathbf{Z}_2,$$

we have

$$\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}, \quad \mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}$$

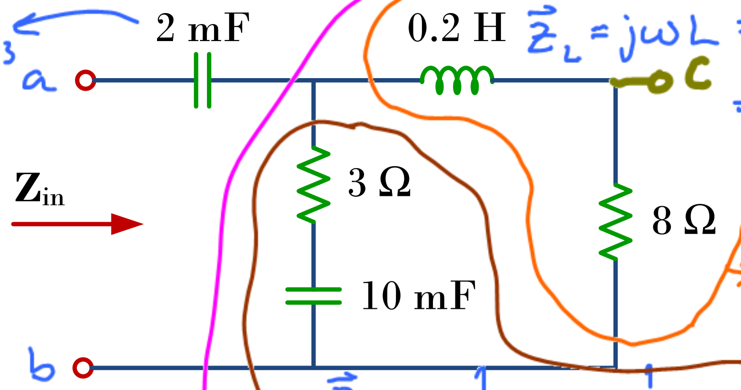
which is the **current-division** principle.



In general, $\vec{I}_k = \dfrac{\frac{1}{\vec{Z}_k}}{\frac{1}{\vec{Z}_1} + \frac{1}{\vec{Z}_2} + \cdots + \frac{1}{\vec{Z}_N}}\,\vec{I}$

EXAMPLE 7.6.1. Find the input impedance of the circuit below. Assume that the circuit operates at $\omega = 50$ rad/s.

at terminals a-b



$\frac{1}{j\omega C} = \frac{1}{j \times 50 \times 2 \times 10^{-3}} = -10j$

$\vec{Z}_L = j\omega L = j \times 50 \times 0.2 = 10j\ \Omega$

$\to 8 + 10j\ \Omega$

$\vec{Z}_C = \frac{1}{j\omega C} = \frac{1}{j \times 50 \times 10 \times 10^{-3}} = -2j\ \Omega$

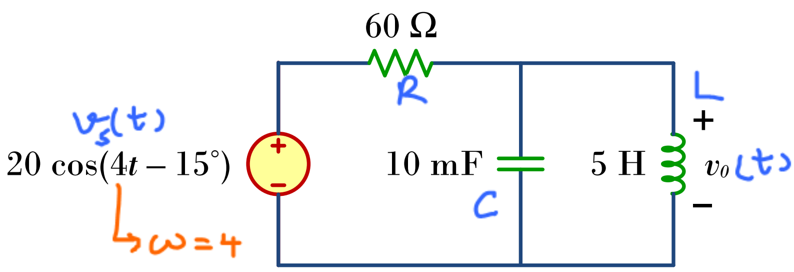
$\to 3 - 2j\ \Omega$

$\vec{Z}_{in} = -10j + \frac{1}{\frac{1}{8+10j} + \frac{1}{3-2j}}$

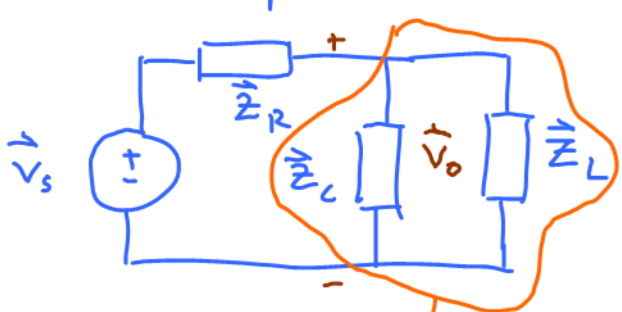
$\frac{1}{\frac{1}{8+10j} + \frac{1}{3-2j}} = \frac{(8+10j)(3-2j)}{8+10j+3-2j} = ?$

$= 3.22 - 11.1j\ \Omega$ ← calc.

EXAMPLE 7.6.2. Determine $v_o(t)$ in the circuit below.



step 1: conversion to phasor domain



$\vec{V}_s = 20\angle{-15^\circ}$

$\vec{Z}_R = 60\ \Omega$

$\vec{Z}_C = \frac{1}{j\omega C} = \frac{1}{j \times 4 \times 10 \times 10^{-3}} = -25j$

$\vec{Z}_L = j\omega L = j \times 4 \times 5 = 20j$

$\vec{Z}_C \parallel \vec{Z}_L = \frac{(-25j)(20j)}{-25j+20j} = 100j$

step 2: voltage divider

$\vec{V}_o = \frac{100j}{60+100j} \times 20\angle{-15^\circ} = 17.15\angle 16^\circ$ ← calc.

step 3: conversion back to time domain

$\Leftrightarrow 17.15\cos(4t + 16^\circ) = v_o(t)$